International Journal of
Applied Mathematics,
Electronics and Computers

IJAMEC

ISSN: 2147-8228 www.atscience.org/IJAMEC

Advanced Technology and Science

Original Research Paper

H∞ Control of Continuous-Time Linear Systems Using Dynamic and Static Output Controllers

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Accepted 15th August 2014

Abstract: The paper deals with the problem of the H∞ control of continuous-time linear systems by proper dynamic output controllers of order equal to the plant model order and by static output controller. The design procedure is based on solution of the set of linear matrix inequalities and the matrix equality and ensures the closed-loop asymptotic stability using Lyapunov approach and the H∞ performance, to guarantee the closed-loop system robustness to unknown disturbance. Numerical examples are given to illustrate the design procedures and the relevance of the method as well as to validate the performances of the proposed approach.

Keywords: *Linear systems, static output control, dynamic output controllers, linear matrix inequality, Lyapunov function, bounded real lemma.*

1. Introduction

In practice, online measurements of all state variables of a process are rarely available and since only their observable outputs are accessible for control purposes, the output feedback control laws have to be considered when the state observers are not applicable. Since, really, the system dynamic may be affected by unmeasurable disturbances the H∞ approach is proposed to be used in the static and dynamic output feedback control law design.

The static output feedback problem seems to be one of the most important question in linear control system design (see, e.g., [1, 2, 3, 4] and the reference therein). Because of the importance of this kind control systems considerable attention was dedicated to the study of suitable design methods. Reflecting the fact that the static output feedback stabilization is generally a concave-convex problem [5], the design conditions based on a solution of various mutually coupled matrix equations or coupled linear matrix inequalities (LMI) was discussed in [6] or, more completely, in [7, 8]. However, unfortunately, the output feedback controller synthesis problem is hard to be converted into a feasible task.

Exploiting the approaches which potentially allow convert dynamic output controller synthesis into an LMI optimization problem, LMI computational technique has brought a preferred tool to solve also this design task. Although the obtained formulations are generally non-convex, using some system bounds they can be formulated as a convex problem. An iterative algorithm for designing the linear time-invariant dynamic output controllers of the prescribed structure was presented in [9], formulating the solution as an optimization based on LMIs in which either the Lyapunov matrix or the controller parameter matrix to be designed are alternately regarded as the optimization variables. Another iterative approach noted as convexifying algorithms was proposed in [10], where the artificially introduced convexifying function in each iteration step is reduced to zero to guaranty the feasibility of the original problem. Applying the controller parameter transformation and a mix of performance measures, the non-recursive approach, noted as the multi-objective synthesis of linear dynamic output-feedback controllers is presented in [11], where each objective is formulated relative to a variety of the closed-loop transfer function and more relaxed sufficient conditions are derived in terms of LMIs.

The aim of this paper is not only to generalize the existing works on the design of static and proper dynamic output control, but above all to formulate the new design conditions based on the set LMIs and, as yet, the linear matrix equality (LME). Applying to the multi-input and multi-output linear systems affected by an unknown disturbance, the standard bound on the system properties is considered as square of the H∞ norm of the closed-loop transfer function matrix between the unknown disturbance input and the system output, and convexifying assumptions are solved by modifying the H∞ control problem. The stability of the closed-loop system is ensured by finding a suitable Lyapunov matrix within a resolution of the LMIs and LME structure.

The paper is organized in six sections. Follow after short introduction in Sec. I, the considered structures of the static and dynamic output controllers are presented in Sec. II. The main results are outlined in Sec. III and IV, formulating stability analysis and suitable design methods for the given types of output control by use of LMIs in such a way that the impact of the unknown disturbance on the controlled system does not exceed a specified limit. In Sec. V the numerical example is given in order to discuss the performances and limitations of the proposed design methods and the last section draws concluding remarks.

Throughout the paper, the notations are narrowly standard in such a way that $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denotes the transpose of the vector $\boldsymbol{x}$, matrix $\boldsymbol{X}$, respectively, $\boldsymbol{X} = \boldsymbol{X}^T > 0$ means that $\boldsymbol{X}$ is a symmetric positive definite matrix, *rank* (°) remits the rank of a matrix, *diag* [°] designates a block diagonal matrix, $\rho(\boldsymbol{X})$ notes the eigenvalues spectrum of the matrix $\boldsymbol{X}$, the symbol $\boldsymbol{I}_n$ indicates the n-th order unit matrix, R denotes the set of real numbers, R^n, $R^{n \times n}$ refer to the set of all n-dimensional real vectors and $n \times n$ real matrices.

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Note:This paper has been presented at the International Conference on Advanced Technology&Sciences (ICAT'14) held in Antalya (Turkey), August 12-15, 2014.

2. Problem Formulation

The systems under consideration are continuous-time linear MIMO systems, described in the state-space form by the set of equations

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{V}\boldsymbol{v}(t) \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) \tag{2}$$

where $\boldsymbol{q}(t) \in R^n, \boldsymbol{u}(t) \in R^r, \boldsymbol{y}(t) \in R^m$ are vectors of the system, input and output variables, respectively, $\boldsymbol{v}(t) \in R^p$ is the exogenous input vector and matrices $\boldsymbol{A} \in R^{n\times n}$, $\boldsymbol{B} \in R^{n\times r}$, $\boldsymbol{V} \in R^{n\times p}$, $\boldsymbol{C} \in R^{m\times n}$ are real matrices, provided that $(\boldsymbol{A},\boldsymbol{B})$ is stabilable and $(\boldsymbol{A},\boldsymbol{C})$ is detectable. It is supposed that an exogenous disturbance input is a non-anticipative process $\boldsymbol{v}(t) \in L_2\langle 0,\infty; R^p)$.
It is assumed that the system is stabilized by the linear static output controller

$$\boldsymbol{u}(t) = \boldsymbol{K}\boldsymbol{y}(t) = \boldsymbol{K}\boldsymbol{C}\boldsymbol{q}(t) \tag{3}$$

where $\boldsymbol{K} \in R^{r\times n}$ is unknown real matrix, as well as by the full order time-invariant dynamic output controller

$$\dot{\boldsymbol{p}}(t) = \boldsymbol{J}\boldsymbol{p}(t) + \boldsymbol{L}\boldsymbol{y}(t) \tag{4}$$

$$\boldsymbol{u}(t) = \boldsymbol{M}\boldsymbol{p}(t) + \boldsymbol{N}\boldsymbol{y}(t) \tag{5}$$

where $\boldsymbol{p}(t) \in R^n$ is the vector of the controller state variables, the controller matrix

$$\boldsymbol{K^o} = \begin{bmatrix} \boldsymbol{J} & \boldsymbol{L} \\ \boldsymbol{M} & \boldsymbol{N} \end{bmatrix} \tag{6}$$

$\boldsymbol{K^o} \in R^{(n+r)\times(n+m)}$, has the prescribed structure with respect to the real matrices $\boldsymbol{J} \in R^{n\times n}$, $\boldsymbol{L} \in R^{n\times m}$, $\boldsymbol{M} \in R^{r\times n}$, $\boldsymbol{N} \in R^{r\times m}$ to be designed.
In the sense of the H∞ control theory, the objective is to design the above given control laws matrix parameters so that, if considering in the control only the measured variable output vector $\boldsymbol{y}(t)$, the impact of the disturbance $\boldsymbol{v}(t)$ on $\boldsymbol{y}(t)$ expressed in term of square of the H∞ norm of the closed-loop transfer function matrix between the disturbance input and the system output, does not exceed a specified limit defined as the guaranteed quadratic performance.

3. Static Output Feedback

This section presents the control law parameter design conditions, corresponding to the static output control (3) acting on the system (1), (2) and optimized in the sense of the H∞ norm of the closed-loop transfer function matrix with respect to the input of unknown disturbance.

3.1. Unforced Mode

Combining equations (1), (2) and (3), the following closed-loop state-space model is achieved

$$\dot{\boldsymbol{q}}(t) = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}\boldsymbol{C})\boldsymbol{q}(t) + \boldsymbol{V}\boldsymbol{v}(t) \tag{7}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) \tag{8}$$

where the closed-loop system matrix $\boldsymbol{A}_\mathrm{c}$ is noted as

$$\boldsymbol{A}_\mathrm{c} = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}\boldsymbol{C}) \tag{9}$$

To give a non-iterative procedure combining the set of LMIs and LME, the following proposition is presented.

Proposition 1: (bounded real lemma modification reflecting the static output controller structure) The closed-loop system, formed by the plant (1), (2) and the static output controller (3), is stable with the quadratic performance γ if there exist a positive definite symmetric matrix Q $\in R^{n\times n}$, a regular matrix H $\in R^{m\times m}$, a matrix Y $\in R^{r\times m}$ and a positive scalar γ $\in R$ such that

$$\boldsymbol{Q} = \boldsymbol{Q}^T > 0, \qquad \gamma > 0 \tag{10}$$

$$\begin{bmatrix} \boldsymbol{A}\boldsymbol{Q} + \boldsymbol{Q}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{Y}\boldsymbol{C} + \boldsymbol{C}^T\boldsymbol{Y}^T\boldsymbol{B}^T & * & * \\ \boldsymbol{V}^T & -\gamma \boldsymbol{I}_p & * \\ \boldsymbol{C}\boldsymbol{Q} & \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{11}$$

$$\boldsymbol{C}\boldsymbol{Q} = \boldsymbol{H}\boldsymbol{C} \tag{12}$$

When the above conditions hold the gain matrices of the static output control law is given by the formula

$$\boldsymbol{K} = \boldsymbol{Y}\boldsymbol{H}^{-1} \tag{13}$$

Here and hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof: (compare, e.g. [1]) Defining the Lyapunov function as follows

$$v(\boldsymbol{q}(t)) = \boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \\ + \int_0^t (\boldsymbol{y^T}(x)\boldsymbol{y}(x) - \gamma\, \boldsymbol{v^T}(x)\boldsymbol{v}(x))\mathrm{d}x > 0 \tag{14}$$

where $\boldsymbol{P} = \boldsymbol{P} > 0$ is a positive definite symmetric matrix and $\sqrt{\gamma} > 0$ is the upper bound of the H∞ norm of the closed-loop system transfer function between the disturbance input and the system output, then [12]

$$\dot{v}(\boldsymbol{q}(t)) = \dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t) + \\ + \boldsymbol{y^T}(t)\boldsymbol{y}(t) - \gamma \boldsymbol{v^T}(t)\boldsymbol{v}(t) < 0 \tag{15}$$

Substituting (7), (8) in (15) gives

$$\dot{v}(\boldsymbol{q}(t)) = \boldsymbol{q}^T(t)(\boldsymbol{A}_c^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_\mathrm{c})\boldsymbol{q}(t) + \\ + \boldsymbol{v^T}(t)\boldsymbol{V}^T\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{V}\boldsymbol{q}(t) + \\ + \boldsymbol{q^T}(t)\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{q}(t) - \gamma \boldsymbol{v^T}(t)\boldsymbol{v}(t) < 0 \tag{16}$$

and with the composed vector

$$\boldsymbol{q}_c^T(t) = [\boldsymbol{q}^T(t) \quad \boldsymbol{v^T}(t)] \tag{17}$$

(16) can be written as

$$\dot{v}(\boldsymbol{q}(t)) = \boldsymbol{q}^T(t)\boldsymbol{P}_\mathrm{c}\, \boldsymbol{q}(t) < 0 \tag{18}$$

where

$$\boldsymbol{P}_\mathrm{c} = \begin{bmatrix} \boldsymbol{A}_c^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_\mathrm{c} + \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{P}\boldsymbol{V} \\ \boldsymbol{V}^T\boldsymbol{P} & -\gamma \boldsymbol{I}_p \end{bmatrix} < 0 \tag{19}$$

Defining the transform matrix

$$\boldsymbol{T} = \mathrm{diag}\,[\boldsymbol{Q} \quad \boldsymbol{I}_p], \qquad \boldsymbol{Q} = \boldsymbol{P^{-1}} \tag{20}$$

pre-multiplying the left-hand side and subsequently the right-hand side of (19) by (20) it can obtain

$$\begin{bmatrix} QA_c^T + A_cQ + QC^TCQ & V \\ V^T & -\gamma I_p \end{bmatrix} < 0 \tag{21}$$

and using the Schur's complement property then the inequality (21) implies

$$\begin{bmatrix} QA_c^T + A_cQ & V & QC^T \\ V^T & -\gamma I_p & \mathbf{0} \\ CQ & \mathbf{0} & -I_m \end{bmatrix} < 0 \tag{22}$$

It has been developed a rule-based FES that made use of the Analysing the matrix element in the upper left corner of (22), i.e.,

$$QA_c^T + A_cQ = Q(A + BKC)^T + (A + BKC)Q \tag{23}$$

it can set

$$BKCQ = BKHH^{-1}CQ \tag{24}$$

where H is a regular square matrix of appropriate dimension.

Defining the following equality

$$H^{-1}C = CQ^{-1} \tag{25}$$

and using the notation

$$Y = KH \tag{26}$$

then

$$BKCQ = BYC \tag{27}$$

$$\begin{aligned} QA_c^T + A_cQ &= Q(A + BKC)^T + (A + BKC)Q = \\ &= AQ + QA^T + BYC + C^TY^TB^T \end{aligned} \tag{28}$$

and (25) implies

$$CQ = HC \tag{29}$$

Now, evidently, (22) with (28) and (29) imply (11), (12), respectively. This concludes the proof. ■

This proposition provides the sufficient condition under LMIs and LME formulations for the synthesis of a static output controller robust to the disturbance $v(t)$.

3.2. Forced Regime

In practice, the case with $r = m$ (square plants) is often encountered, where it is generally associated with each output signal a reference signal, which is expected to influence as wanted this output. Such regime, reflecting nonzero set working points, is called the forced regime.

Definition 1: The forced regime for (1), (2) with the static output control is given by the control policy

$$u(t) = KCq(t) + Ww(t) \tag{30}$$

where $r = m$, $w(t) \in R^m$ is desired output signal vector, and matrix $W \in R^{m \times m}$ is the signal gain matrix.

Proposition 2: If the system (1), (2) is stabilizable by the control policy (30), and [13]

$$\operatorname{rank}\begin{bmatrix} A & B \\ C & \mathbf{0} \end{bmatrix} = n + m \tag{31}$$

then the matrix W in (30), designed by using the static decoupling principle, takes the form

$$W = -(C(A + BKC)^{-1}B)^{-1} \tag{32}$$

Proof: In a steady state which corresponds to $\dot{q}(t) = \mathbf{0}$, the equality $y_o = w_o$ must hold, where $q_o \in R^n$, $y_o, w_o \in R^m$ are vectors of steady state values of $q(t)$, $y(t)$, and $w(t)$.

In this case, the disturbance-free system equations (1), (2) under control (30) implies

$$0 = (A + BKC)q_o + BWw_o \tag{33}$$

$$y_o = Cq_o = -C(A + BKC)^{-1}BWw_o = -I_m w_o \tag{34}$$

It is evident that the equality (34) implies (32). This concludes the proof. ■

The W matrix is nothing else than the inverse of the closed-loop static gain matrix. This gain matrix can be obtained so by setting $s = 0$ in the state-space expression of the transfer function matrix of the closed-loop system with respect to the forced input. Note, the static gain realized by the W matrix is ideal in control only if the plant parameters, on which the value of W depends, are known and do not vary with time. The forced regime is basically designed for constant references and is very closely related to shift of origin. If the command value $w(t)$ is changed "slowly enough," the above scheme can do a reasonable job of tracking, i.e., making $y(t)$ to follow $w(t)$ [14].

4. Dynamic Output Feedback

In this part the time-invariant dynamic output controller is considered in control of the linear continuous-time system perturbed by an unknown input disturbance.

4.1. Unforced Mode

To analyse the stability of the closed-loop system structure with the dynamic output controller, the following form of the system description can be introduced.

$$\begin{bmatrix} \dot{q}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A + BNC & BM \\ LC & J \end{bmatrix} \begin{bmatrix} q(t) \\ p(t) \end{bmatrix} + \begin{bmatrix} V \\ \mathbf{0} \end{bmatrix} v(t) \tag{35}$$

$$y(t) = \begin{bmatrix} \mathbf{0} & I_m \end{bmatrix} \begin{bmatrix} \mathbf{0} & I_n \\ C & \mathbf{0} \end{bmatrix} \begin{bmatrix} q(t) \\ p(t) \end{bmatrix} \tag{36}$$

After introducing the notations

$$q^{\circ T}(t) = \begin{bmatrix} q^T(t) & p^T(t) \end{bmatrix} \tag{37}$$

$$A^\circ = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad B^\circ = \begin{bmatrix} \mathbf{0} & B \\ I_n & \mathbf{0} \end{bmatrix}, \quad C^\circ = \begin{bmatrix} \mathbf{0} & I_n \\ C & \mathbf{0} \end{bmatrix} \tag{38}$$

$$V^{\circ T} = \begin{bmatrix} V^\circ & \mathbf{0} \end{bmatrix}, \qquad I^\circ = \begin{bmatrix} \mathbf{0} & I_m \end{bmatrix} \tag{39}$$

where $A^\circ \in R^{2n \times 2n}$, $B^\circ \in R^{2n \times (n+r)}$, $C^\circ \in R^{(n+m) \times 2n}$, $V^\circ \in R^{2n \times p}$, $I \in R^{m \times (n+m)}$, the closed-loop state-space dynamic equations as well as the system output relation takes the following form

$$\dot{q}^\circ(t) = (A^\circ + B^\circ K^\circ C^\circ)q^\circ(t) + V^\circ v(t) \tag{40}$$

$$y^\circ(t) = I^\circ C^\circ q^\circ(t) \tag{41}$$

respectively.

In the sequel, so it is supposed that (A°, B°) is stabilable and (A°, C°) is detectable [15], [16] and the matrix product $C^\circ B^\circ$ is nonzero matrix.

Theorem 1: (bounded real lemma modification reflecting the dynamic output controller structure) The closed-loop system consisting of the plant (1), (2) with the controller (4), (5) is stable with the quadratic performance γ if there exist a symmetric positive definite matrix $\boldsymbol{Q}^{\mathrm{o}} \in R^{2n\times 2n}$, a regular matrix $\boldsymbol{H}^{\mathrm{o}} \in R^{(n+m)\times(n+m)}$, a matrix $\boldsymbol{Y}^{\mathrm{o}} \in R^{(n+r)\times(n+m)}$ and a positive $\gamma \in R$ such that

$$\boldsymbol{Q}^{\mathrm{o}} = \boldsymbol{Q}^{\mathrm{o}T} > 0, \qquad \gamma > 0 \tag{42}$$

$$\begin{bmatrix} \boldsymbol{Z}^{\mathrm{o}} & * & * \\ \boldsymbol{V}^{\mathrm{o}T} & -\gamma \boldsymbol{I}_p & * \\ \boldsymbol{I}^{\mathrm{o}}\boldsymbol{C}^{\mathrm{o}}\mathrm{Q}^{\boldsymbol{o}} & \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{43}$$

$$\boldsymbol{C}^{\mathrm{o}}\boldsymbol{Q}^{\mathrm{o}} = \boldsymbol{H}^{\mathrm{o}}\boldsymbol{C}^{\mathrm{o}} \tag{44}$$

where

$$\boldsymbol{Z}^{\mathrm{o}} = \boldsymbol{A}^{\mathrm{o}}\boldsymbol{Q}^{\mathrm{o}} + \boldsymbol{Q}^{\mathrm{o}}\boldsymbol{A}^{\mathrm{o}T} + \boldsymbol{B}^{\mathrm{o}}\boldsymbol{Y}^{\mathrm{o}}\boldsymbol{C}^{\mathrm{o}} + \boldsymbol{C}^{\mathrm{o}T}\boldsymbol{Y}^{\mathrm{o}T}\boldsymbol{B}^{\mathrm{o}T} \tag{45}$$

When the above conditions hold the common control law gain matrix of the dynamic output controller is given by the following formula

$$\boldsymbol{K}^{\mathrm{o}} = \boldsymbol{Y}^{\mathrm{o}}(\boldsymbol{H}^{\mathrm{o}})^{-\boldsymbol{1}} \tag{46}$$

Proof: Defining the Lyapunov function as follows

$$v\big(\boldsymbol{q}^{\mathrm{o}}(t)\big) = \boldsymbol{q}^{\mathrm{o}T}(t)\boldsymbol{P}^{\mathrm{o}}\boldsymbol{q}^{\mathrm{o}}(t) + \\ + \int_0^t (\boldsymbol{y}^{\boldsymbol{T}}(x)\boldsymbol{y}(x) - \gamma\, \boldsymbol{v}^{\boldsymbol{T}}(x)\boldsymbol{v}(x))\mathrm{d}x > 0 \tag{47}$$

where $\boldsymbol{P}^{\mathrm{o}} = \boldsymbol{P}^{\mathrm{o}T} > \boldsymbol{0}$ is a positive definite symmetric matrix and $\sqrt{\gamma} > 0$ is the upper bound of the H_∞ norm of the closed-loop system transfer function between the disturbance input and the system output, then

$$\dot{v}\big(\boldsymbol{q}^{\mathrm{o}}(t)\big) = \dot{\boldsymbol{q}}^{\mathrm{o}T}(t)\boldsymbol{P}^{\mathrm{o}}\boldsymbol{q}^{\mathrm{o}}(t) + \boldsymbol{q}^{\mathrm{o}T}(t)\boldsymbol{P}^{\mathrm{o}}\dot{\boldsymbol{q}}^{\mathrm{o}}(t) + \\ +\boldsymbol{y}^{\boldsymbol{T}}(t)\boldsymbol{y}(t) - \gamma\boldsymbol{v}^{\boldsymbol{T}}(t)\boldsymbol{v}(t) < 0 \tag{48}$$

Substituting (40) and (41) into (48) it yields

$$\dot{v}\big(\boldsymbol{q}^{\mathrm{o}}(t)\big) = \boldsymbol{q}^{\mathrm{o}T}(t)(\boldsymbol{A}_{\boldsymbol{c}}^{\mathrm{o}T}\boldsymbol{P}^{\mathrm{o}} + \boldsymbol{P}^{\mathrm{o}}\boldsymbol{A}_{\boldsymbol{c}}^{\mathrm{o}})\boldsymbol{q}^{\mathrm{o}}(t) + \\ +\boldsymbol{v}^{\mathrm{o}T}(t)\boldsymbol{V}^{\mathrm{o}T}\boldsymbol{P}^{\mathrm{o}}\boldsymbol{q}(t) + \boldsymbol{q}^{\mathrm{o}T}(t)\boldsymbol{P}^{\mathrm{o}}\boldsymbol{V}^{\mathrm{o}}\boldsymbol{v}^{\mathrm{o}}(t) + \\ +\boldsymbol{q}^{\mathrm{o}T}(t)\boldsymbol{C}^{\mathrm{o}T}\boldsymbol{I}^{\mathrm{o}T}\boldsymbol{I}^{\mathrm{o}}\boldsymbol{C}^{\mathrm{o}}\boldsymbol{q}^{\mathrm{o}}(t) - \gamma\boldsymbol{v}^{\boldsymbol{T}}(t)\boldsymbol{v}(t) < 0 \tag{49}$$

where $\boldsymbol{A}_{\boldsymbol{c}}^{\mathrm{o}} = \boldsymbol{A}^{\mathrm{o}} + \boldsymbol{B}^{\mathrm{o}}\boldsymbol{K}^{\mathrm{o}}\boldsymbol{C}^{\mathrm{o}}$**.** Then with the composed vector

$$\boldsymbol{q}_{\boldsymbol{c}}^{\mathrm{o}T}(t) = [\boldsymbol{q}^{\mathrm{o}T}(t) \quad \boldsymbol{v}^{\boldsymbol{T}}(t)] \tag{50}$$

the inequality (49) can be written as

$$\dot{v}\big(\boldsymbol{q}_{\boldsymbol{c}}^{\mathrm{o}}(t)\big) = \boldsymbol{q}_{\boldsymbol{c}}^{\mathrm{o}T}(t)\boldsymbol{P}_{\boldsymbol{c}}^{\mathrm{o}}\,\boldsymbol{q}_{\boldsymbol{c}}^{\mathrm{o}}(t) < 0 \tag{51}$$

where

$$\boldsymbol{P}_{\boldsymbol{c}}^{\mathrm{o}} = \begin{bmatrix} \boldsymbol{A}_{\boldsymbol{c}}^{\mathrm{o}T}\boldsymbol{P}^{\mathrm{o}} + \boldsymbol{P}^{\mathrm{o}}\boldsymbol{A}_{\boldsymbol{c}}^{\mathrm{o}} + \boldsymbol{C}^{\mathrm{o}T}\boldsymbol{I}^{\mathrm{o}T}\boldsymbol{I}^{\mathrm{o}}\boldsymbol{C}^{\mathrm{o}} & \boldsymbol{P}^{\mathrm{o}}\boldsymbol{V}^{\mathrm{o}} \\ \boldsymbol{V}^{\mathrm{o}T}\boldsymbol{P}^{\mathrm{o}} & -\gamma\boldsymbol{I}_p \end{bmatrix} < 0 \tag{52}$$

Defining the transform matrix

$$\boldsymbol{T}^{\mathrm{o}} = \mathrm{diag}\,[\boldsymbol{Q}^{\mathrm{o}} \quad \boldsymbol{I}_p], \qquad \boldsymbol{Q}^{\mathrm{o}} = (\boldsymbol{P}^{\mathrm{o}})^{-\boldsymbol{1}} \tag{53}$$

pre-multiplying the left-hand side and subsequently the right-hand side of (52) by (53) it can obtain

$$\begin{bmatrix} \boldsymbol{Q}^{\mathrm{o}}\boldsymbol{A}_{\boldsymbol{c}}^{\mathrm{o}T} + \boldsymbol{A}_{\boldsymbol{c}}^{\mathrm{o}}\boldsymbol{Q} + \boldsymbol{Q}^{\mathrm{o}}\boldsymbol{C}^{\mathrm{o}T}\boldsymbol{I}^{\mathrm{o}T}\boldsymbol{I}^{\mathrm{o}}\boldsymbol{C}^{\mathrm{o}}\boldsymbol{Q}^{\mathrm{o}} & \boldsymbol{V}^{\mathrm{o}} \\ \boldsymbol{V}^{\mathrm{o}T} & -\gamma\boldsymbol{I}_p \end{bmatrix} < 0 \tag{54}$$

Thus, with the Schur complement property, (54) can be rewritten as

$$\begin{bmatrix} \boldsymbol{Q}^{\mathrm{o}}\boldsymbol{A}_{\boldsymbol{c}}^{\mathrm{o}T} + \boldsymbol{A}_{\boldsymbol{c}}^{\mathrm{o}}\boldsymbol{Q}^{\mathrm{o}} & \boldsymbol{V}^{\mathrm{o}} & \boldsymbol{Q}^{\mathrm{o}}\boldsymbol{C}^{\mathrm{o}T}\boldsymbol{I}^{\mathrm{o}T} \\ \boldsymbol{V}^{\mathrm{o}T} & -\gamma\boldsymbol{I}_p & \boldsymbol{0} \\ \boldsymbol{I}^{\mathrm{o}}\boldsymbol{C}^{\mathrm{o}}\boldsymbol{Q}^{\mathrm{o}} & \boldsymbol{0} & -\boldsymbol{I}_{\boldsymbol{m}} \end{bmatrix} < 0 \tag{55}$$

Analysing the matrix element in the upper left corner of (55), i.e.,

$$\boldsymbol{Q}^{\mathrm{o}}\boldsymbol{A}_{\boldsymbol{c}}^{\mathrm{o}T} + \boldsymbol{A}_{\boldsymbol{c}}^{\mathrm{o}}\boldsymbol{Q}^{\mathrm{o}} = \\ \boldsymbol{Q}^{\mathrm{o}}(\boldsymbol{A}^{\mathrm{o}} + \boldsymbol{B}^{\mathrm{o}}\boldsymbol{K}^{\mathrm{o}}\boldsymbol{C}^{\mathrm{o}})^T + (\boldsymbol{A}^{\mathrm{o}} + \boldsymbol{B}^{\mathrm{o}}\boldsymbol{K}^{\mathrm{o}}\boldsymbol{C}^{\mathrm{o}})\boldsymbol{Q}^{\mathrm{o}} \tag{56}$$

then analogously with (24)-(26) it can be set

$$\boldsymbol{Z}^{\mathrm{o}} = \boldsymbol{A}^{\mathrm{o}}\boldsymbol{Q}^{\mathrm{o}} + \boldsymbol{Q}^{\mathrm{o}}\boldsymbol{A}^{\mathrm{o}T} + \boldsymbol{B}^{\mathrm{o}}\boldsymbol{Y}^{\mathrm{o}}\boldsymbol{C}^{\mathrm{o}} + \boldsymbol{C}^{\mathrm{o}T}\boldsymbol{Y}^{\mathrm{o}T}\boldsymbol{B}^{\mathrm{o}T} \tag{57}$$

and with the notations

$$\boldsymbol{C}^{\mathrm{o}}\boldsymbol{Q}^{\mathrm{o}} = \boldsymbol{H}^{\mathrm{o}}\boldsymbol{C}^{\mathrm{o}} \tag{58}$$

$$\boldsymbol{Y}^{\mathrm{o}} = \boldsymbol{K}^{\mathrm{o}}\boldsymbol{H}^{\mathrm{o}} \tag{59}$$

(55) with (57) imply (43), and (58) gives (44). This concludes the proof. ■

Forced Regime

Definition 2: The forced regime for (1), (2) with the dynamic output controller is given by the control policy

$$\dot{\boldsymbol{p}}(t) = \boldsymbol{Jp}(t) + \boldsymbol{Ly}(t) \tag{60}$$

$$\boldsymbol{u}(t) = \boldsymbol{Mp}(t) + \boldsymbol{Ny}(t) + \boldsymbol{Ww}(t) \tag{61}$$

where $r = m$, $\boldsymbol{w}(t) \in R^m$ is desired output signal vector, and matrix $\boldsymbol{W} \in R^{m\times m}$ is the signal gain matrix.

Theorem 2: If the system (1), (2) is stabilizable by the control policy (60), (61), and if (31) is satisfied, then the matrix signal gain matrix in (61), designed by using the static decoupling principle, takes the form

$$\boldsymbol{W} = -(\boldsymbol{C}(\boldsymbol{A} - \boldsymbol{BMJ}^{-\boldsymbol{1}}\boldsymbol{LC} + \boldsymbol{BNC})^{-\boldsymbol{1}}\boldsymbol{B})^{-\boldsymbol{1}}$$

Proof: In a steady state the disturbance-free system equations (1), (2) under control (60), (61) imply

$$0 = \boldsymbol{Aq}_{\mathrm{o}} + \boldsymbol{Bu}_{\mathrm{o}}$$

$$0 = \boldsymbol{Jp}_{\mathrm{o}} + \boldsymbol{LCq}_{\mathrm{o}}$$

Since now (61), (64) implies

$$\boldsymbol{u}_{\mathrm{o}} = \big(-\boldsymbol{MJ}^{-\boldsymbol{1}}\boldsymbol{LC} + \boldsymbol{NC}\big)\boldsymbol{q}_{\mathrm{o}} + \boldsymbol{Ww}_{\mathrm{o}}$$

then, substituting (65) into (63) it yields

$$\boldsymbol{0} = \big(\boldsymbol{A} - \boldsymbol{BMJ}^{-\boldsymbol{1}}\boldsymbol{LC} + \boldsymbol{BNC}\big)\boldsymbol{q}_{\mathrm{o}} + \boldsymbol{BWw}_{\mathrm{o}}$$

$$\boldsymbol{q}_{\mathrm{o}} = -\big(\boldsymbol{A} - \boldsymbol{BMJ}^{-\boldsymbol{1}}\boldsymbol{LC} + \boldsymbol{BNC}\big)^{-1}\boldsymbol{BWw}_{\mathrm{o}}$$

respectively, and

$$\boldsymbol{y}_{\mathrm{o}} = \boldsymbol{Cq}_{\mathrm{o}} = -\boldsymbol{C}\big(\boldsymbol{A} - \boldsymbol{BMJ}^{-\boldsymbol{1}}\boldsymbol{LC} + \boldsymbol{BNC}\big)^{-1}\boldsymbol{BWw}_{\mathrm{o}}$$

Thus, considering $\boldsymbol{y}_{\mathrm{o}} = \boldsymbol{w}_{\mathrm{o}}$ then (68) implies (62). This concludes the proof. ■

5. Illustrative Example

The features of the considered schemes and the effectiveness of the proposed design conditions are presented using the illustrative example.
The state space representation, describing the chemical reactor model [17], consists of the following matrices

$$\boldsymbol{A} = \begin{bmatrix} 1.380 & -2.080 & 6.715 & -5.676 \\ -0.581 & -4.290 & 0.000 & 0.675 \\ 10.672 & 4.273 & -6.654 & 5.893 \\ 0.482 & 4.273 & 1.343 & -2.104 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} 0.000 & 0.000 \\ 5.679 & 0.000 \\ 1.136 & -3.146 \\ 1.136 & 0.000 \end{bmatrix}, \quad \boldsymbol{V} = \begin{bmatrix} 0.000 \\ 3.397 \\ 9.872 \\ 0.000 \end{bmatrix}$$

$$\mathrm{C} = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and this system in the considered closed loop structures under the static output feedback (3), as well as under the dynamic output feedback (4), (5) was used in the presented simulations.
Note, the pair provided that $(\boldsymbol{A}, , \boldsymbol{B})$ is controllable and $(\boldsymbol{A}, \boldsymbol{C})$ is observable.
Within the above system parameters, solving the LMIs (10)-(12) using the SeDuMi package for MATLAB [18], the following quadratic performance and the control law matrix parameter were obtained
$\gamma = 10.3628$

$$\boldsymbol{K} = \begin{bmatrix} 0.4683 & -4.0758 \\ 4.6458 & -5.2372 \end{bmatrix}$$

with the closed-loop system matrix eigenvalues spectrum

$$\rho(\boldsymbol{A}_c) = \{-0.1062 \pm 0.1543i, -0.0457 \pm 0.0920i$$

guaranteeing closed-loop system stability.

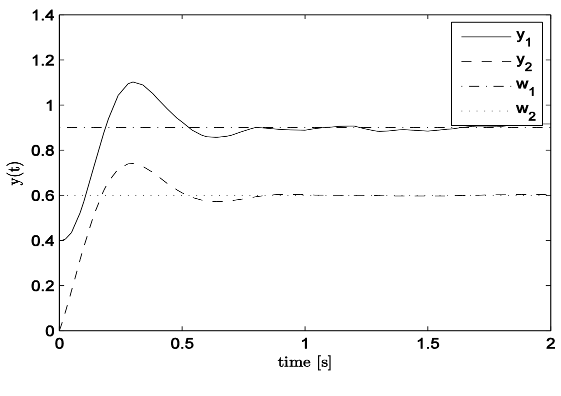

Figure 1. System output response for static output controller.

Solving (42)-(44) for $\boldsymbol{Q}^{\mathrm{o}}, \boldsymbol{Y}^{\mathrm{o}}, \boldsymbol{H}^{\mathrm{o}}$ using the same solver SeDuMi, the quadratic performance upper-bound and the dynamic output controller matrix parameters were as follows

$$\gamma = 10.3628$$

$$\boldsymbol{J} = \begin{bmatrix} -0.7086 & 0.0002 & 0.0009 & -0.0003 \\ 0.0002 & -0.7106 & 0.0003 & 0.0000 \\ 0.0009 & 0.0003 & -0.7108 & 0.0003 \\ -0.0003 & 0.0000 & 0.0003 & -0.7085 \end{bmatrix}$$

$$\mathrm{M} = \begin{bmatrix} -0.0002 & -0.0002 & -0.0003 & 0.0003 \\ -0.0010 & 0.0003 & -0.0005 & 0.0011 \end{bmatrix}$$

$$\boldsymbol{N} = \begin{bmatrix} 0.4748 & -4.1352 \\ 4.4644 & -5.0116 \end{bmatrix}$$

$$\mathrm{L} = \begin{bmatrix} -0.0010 & 0.0013 \\ 0.0005 & 0.0012 \\ 0.0007 & 0.0047 \\ 0.0015 & -0.0056 \end{bmatrix}$$

Now, the resulting global closed-loop system eigenvalues spectrum was

$$\rho(\boldsymbol{A}_c^o) = \begin{Bmatrix} -0.1034 \pm 0.1516\mathrm{i}, -0.0459 \pm 0.0927\mathrm{i} \\ -0.0071, -0.0071, -0.0071, -0.0701 \end{Bmatrix}$$

Since the system matrix $\boldsymbol{J}$ of the dynamic output controller is block-diagonally dominant with the stable real eigenvalue spectrum, this, in turn, means that the remaining eigenvalues of the system matrix $\boldsymbol{A}_c^o$ of the closed-loop system with the dynamic controller are approximately the same as the eigenvalues of the system matrix $\boldsymbol{A}_c$ of the closed-loop system with the static output controller.
The both static as well as dynamic controller design methods previously described were applied to the simulation benchmark. The conditions in simulations were specified for system in the forced regimes, where

$$\boldsymbol{q}^T(0) = [0.1 \quad 0.0 \quad 0.0 \quad 0.0], \ \boldsymbol{w}^T(0) = [0.9 \quad 0.6]$$

$$p^T(0) = \boldsymbol{0}, \qquad \sigma_v^2 = 0.02$$

and the signal gain matrices $\boldsymbol{W}_s, \boldsymbol{W}_d$ were computed using (32), (62), respectively, as follows

$$\boldsymbol{W}_s = \begin{bmatrix} -0.4370 & 4.0915 \\ -4.4232 & 5.4242 \end{bmatrix}, \boldsymbol{W}_d = \begin{bmatrix} -0.4435 & 4.1510 \\ -4.4218 & 5.1985 \end{bmatrix}$$

Since the same desired output variables have been utilized to assess the each controller ability response and to demonstrate performance with respect to asymptotic properties, the results of the both proposed design method can be immediately compared.

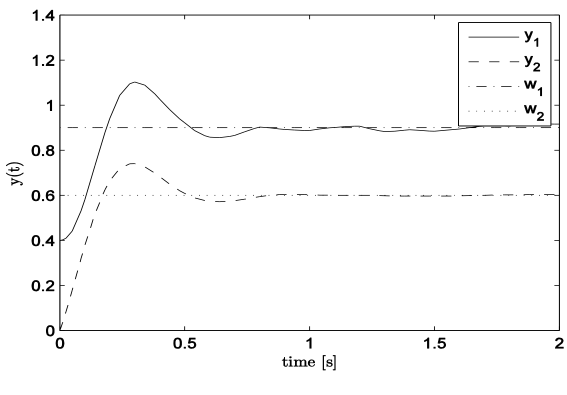

Figure 2. System output response for dynamic output controller.

Fig. 1 shows the closed-loop system output response with the static output controller (3) which matrix parameter was obtained solving (10)-(12). Using the dynamic output controller (4), (5) with the gain matrix parameters satisfying the conditions (42)-(44), Fig. 2 shows the output system response of the closed-loop system for the same system initial conditions and the control policy (60), (61).

It is obvious from these figures that both controllers which parameters were obtained using the solutions of the LMI problems specified by Proposition 1 and Theorem 1 can successfully provide for the closed-loop system steady-state properties and asymptotic dynamics. Since the design conditions guarantee optimization in the sense of H_∞ norm of the both closed-loop disturbance transfer functions, although the dynamic controller provides slightly smaller value of this norm, there is practically no difference between the output responses.

Using the recursive methods of dynamic controller's synthesis, these differences can be significant because the obtained matrix parameter values of the static and dynamic controller could be affected by thresholds at which the iterative design process is completed.

6. Concluding Remarks

New approach for output dynamic feedback control design is presented in this paper. By the proposed procedure the control problem is parameterized in such LMIs set with one additional LME which admit more freedom in guaranteeing the output feedback control quadratic performance with respect to unknown disturbance acting on the system. Sufficient conditions of the controller existence manipulating the stability of the closed-loop systems imply the control structure, which stabilize the system in the sense of Lyapunov and the controller design tasks is a solvable numerical problem. An additional benefit of the method is that controller uses minimum feedback information with respect to desired system output and the approach is enough flexible to allow the inclusion of additional design condition bounds. Of course, it cannot find output feedback controllers to guarantee the stability of the closed-loop system if uncontrolled modes are unstable although a common Lyapunov function exists in the feasible design conditions.

Acknowledgements

The work presented in the paper was supported by VEGA, the Grant Agency of the Ministry of Education and the Academy of Science of Slovak Republic, under Grant No. 1/0348/14. This support is very gratefully acknowledged.

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