International Journal of
Applied Mathematics,
Electronics and Computers
IJAMEC
ISSN: 2147-8228 http://ijamec.atscience.org
Advanced Technology and Science
Original Research Paper

A Study on the Damped Free Vibration with Fractional Calculus

Okkeş Ozturk*[1]

Accepted 05th May 2016

Abstract: Fractional calculus theory includes the definitions of derivatives and integrals with arbitrary order. This theory is used to solve some classes of differential equations and fractional differential equations. One of these equations is the damped free vibration equation. This equation is a linear homogeneous differential equation with constant coefficients. In this paper, we intend to solve this equation by means of N-fractional calculus method.

Keywords: *Fractional calculus, Differintegral, Ordinary differential equations, Generalized Leibniz rule, N-Fractional calculus method, The damped free vibration equation.*

1. Introduction

The fractional calculus theory enables a set of axioms and methods to generalize the coordinate and corresponding derivative notions from integer n to arbitrary order μ, $\{t^n, \partial^n/\partial t^n\} \rightarrow \{t^\mu, \partial^\mu/\partial t^\mu\}$ under favorable conditions. Fractional differential equations are applied in a widespread manner in robot technology, PID control systems, Schrödinger equation, heat transfer, relativity theory, economy, filtration, controller design, mechanics, optics, modelling and so on [1,2].

Recently, by applying the fractional calculus definitions of a differintegral (*that is, fractional derivative and fractional integral*) of order $\mu \in \mathbb{R}$, many authors have explicity obtained particular solutions of a number of families of homogeneous *(as well as non-homogeneous)* linear ordinary and partial differintegral equations [3-5]. An important example of Fuchsian differential equations is provided by the celebrated hypergeometric equation *(or, more precisely, the Gauss hypergeometric equation)*

$$z(1-z)\frac{d^2u}{dz^2} + [\gamma - (a+b+1)z]\frac{du}{dz} - abu = 0,$$

whose study can be traced back to L. Euler, C. F. Gauss and E. E. Kummer. On the other hand, a special limit *(confluent)* case of the Gauss hypergeometric equation, in the form [6]

$$\frac{d^2u}{dz^2} + \left(-\frac{1}{4} + \frac{\varkappa}{z} - \frac{\ell(\ell+1)}{z^2}\right)u = 0,$$

is refered to as the Whittaker equation whose systematic study was initiated by E. T. Whittaker.

Other classes of non-Fuchsian differential equations which we shall consider in this investigation include the so-called Fukuhara equation [7]

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} - (1 - z + z^2)u = 0,$$

the Tricomi equation [8]

$$\frac{d^2u}{dz^2} + \left(\alpha + \frac{\beta}{z}\right)\frac{du}{dz} + \left(\gamma + \frac{\delta}{z} + \frac{\varepsilon}{z^2}\right)u = 0,$$

and the Bessel equation [9]

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2 - v^2)u = 0.$$

Moreover, Feldman [10] used Hilbert transform for non-linear system vibration analysis. Bagley and Torvik [11] analysed the viscoelastically damped structures by means of fractional calculus. In [12], Gough expressed the nonlinear free vibration of a damped elastic string. Rossikhin and Shitikova [13] applied the fractional derivatives to the analysis of damped vibrations of viscoelastic single mass systems and, they [14] studied on nonlinear damped vibrations of suspension bridges by applying farctional calculus.

2. Preliminaries

Some of most obvious formulations based on the fundamental definitions of Riemann-Liouville fractional differentiation and fractional integration are, respectively [1,2],

$$ {}_aD_\tau^\mu f(\tau) = f_\mu(\tau) = \frac{1}{\Gamma(n-\mu)}\frac{d^n}{d\tau^n} \times \int_a^\tau f(t)(\tau - t)^{n-\mu-1}\,dt,$$

$(n - 1 \le \mu < n)$,

and, the integral operator $D^{-\mu}$ as

$$ {}_aD_\tau^{-\mu} f(\tau) = f_{-\mu}(\tau) = \frac{1}{\Gamma(\mu)}\int_a^\tau f(t)(\tau - t)^{\mu-1}\,dt,$$

$(\tau > a, \mu > 0)$,

where $n \in \mathbb{N}$, $D = d/dt$ and $\Gamma(t)$ is the Gamma function

$$\Gamma(t) = \int_0^\infty e^{-z}z^{t-1}\,dz.$$

Definition 2.1. [2] If the function $f(z)$ is analytic (*regular*) inside and on C, where $C = \{C^-, C^+\}$, C^- is a contour along the cut joining the points z and $-\infty + i\mathrm{Im}(z)$, which starts from the point at $-\infty$, encircles the point z once counter-clockwise, and returns to the point at $-\infty$, and C^+ is a contour along the cut

[1] *Department of Mathematics, Faculty of Arts and Sciences, Bitlis Eren University, 13000, Bitlis/Turkey*
* *Corresponding Author: Email: oozturk27@gmail.com*

joining the points z and $\infty + i\mathrm{Im}(z)$, which starts from the point at ∞, encircles the point z once counter-clockwise, and returns to the point at ∞,

$$f_\mu(z) = [f(z)]_\mu = \frac{\Gamma(\mu+1)}{2\pi i}\int_C \frac{f(t)dt}{(t-z)^{\mu+1}} \quad (\mu \notin \mathbb{Z}^-),$$

$$f_{-n}(z) = \lim_{\mu\to -n} f_\mu(z) \quad (n \in \mathbb{Z}^+),$$

where $t \neq z$,

$-\pi \le \arg(t-z) \le \pi$ for C^-,

$0 \le \arg(t-z) \le 2\pi$ for C^+.

In that case, $f_\mu(z)$ $(\mu > 0)$ is the fractional derivative of $f(z)$ of order μ and $f_\mu(z)$ $(\mu < 0)$ is the fractional integral of $f(z)$ of order $-\mu$, confirmed (*in each case*) that

$|f_\mu(z)| < \infty \quad (\mu \in \mathbb{R})$.

Lemma 2.1. *(Linearity)* Let $f(z)$ and $g(z)$ be analytic and single-valued functions. If f_μ and g_μ exist, then

(i) $[Af(z)]_\mu = A[f(z)]_\mu$,

(ii) $[Af(z) + Bg(z)]_\mu = A[f(z)]_\mu + B[g(z)]_\mu$,

where A and B are constants and $\mu \in \mathbb{R}$, $z \in \mathbb{C}$.

Lemma 2.2. *(Index law)* Let $f(z)$ be an analytic and single-valued function. If $(f_\nu)_\mu$ and $(f_\mu)_\nu$ exist, then

$$\{[f(z)]_\nu\}_\mu = [f(z)]_{\nu+\mu} = \{[f(z)]_\mu\}_\nu,$$

where $\nu, \mu \in \mathbb{R}$, $z \in \mathbb{C}$ and $\left|\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+1)\Gamma(\mu+1)}\right| < \infty$.

Property 2.1. For a constant λ,

$$\left(e^{\lambda z}\right)_\mu = \lambda^\mu e^{\lambda z} \quad (\lambda \neq 0, \mu \in \mathbb{R}, z \in \mathbb{C}).$$

Property 2.2. For a constant λ,

$$\left(e^{-\lambda z}\right)_\mu = e^{-i\pi\mu}\lambda^\mu e^{-\lambda z} \quad (\lambda \neq 0, \mu \in \mathbb{R}, z \in \mathbb{C}).$$

Property 2.3. For a constant λ,

$$\left(z^\lambda\right)_\mu = e^{-i\pi\mu} z^{\lambda-\mu}\frac{\Gamma(\mu-\lambda)}{\Gamma(-\lambda)},$$

$$\left(\mu \in \mathbb{R}, z \in \mathbb{C}, \left|\frac{\Gamma(\mu-\lambda)}{\Gamma(-\lambda)}\right| < \infty\right).$$

Lemma 2.3. *(Generalized Leibniz rule)* Let $f(z)$ and $g(z)$ be single-valued and analytic functions. If f_μ and g_μ exist, then

$$N^\mu(f.g) = (f.g)_\mu = \sum_{n=0}^{\infty} \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-n)\Gamma(n+1)} \times f_{\mu-n}.g_n,$$

where $\mu \in \mathbb{R}$, $z \in \mathbb{C}$ and $\left|\frac{\Gamma(\mu+1)}{\Gamma(\mu+1-n)\Gamma(n+1)}\right| < \infty$.

2.1. Main Equation

The damped free vibration equation is given by

$$x_2 + \alpha x_1 + \beta x = 0 \quad x = x(t), \tag{1}$$

where $x_n = d^n x/dt^n$ $(n = 0,1,2,\dots)$ and $\alpha = c/m$, $\beta = k/m$. x is displacement of the mass, t is time, m is mass, c is damping coefficient and k is spring constant in a mass-spring-damper system.

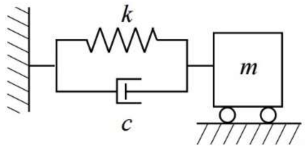

Figure 1. Mass-spring-damper system.

In this study, our aim is that obtain the solutions of Eq. (1) by using N-fractional calculus method. Eq. (1) can already be solved with simpler methods. The purpose of solving the equation via N-fractional calculus method is that show the existence of a different method for this equation. This method is more complex than the other methods. In this paper, we applied this method for a simple equation (Eq. (1)). However, N-fractional calculus method can be used for the complex equations. This method can be generalized from simple to complex. And, we will apply this method to the complex equations in the future time.

3. N-Fractional Calculus Method for Eq. (1)

At first, we obtain

$$t^2 f_2 + (\alpha t^2 + 4t) f_1 + (\beta t^2 + 2\alpha t + 2) f = 0, \tag{2}$$

by using the transformation as

$$x = t^2 f \quad (t \neq 0, f = f(t)).$$

Now, we suppose that

$$f = t^\nu g \quad (t \neq 0, g = g(t)). \tag{3}$$

Thus, we have

$$\begin{aligned} f_1 &= \nu t^{\nu-1} g + t^\nu g_1, \\ f_2 &= \nu(\nu-1)t^{\nu-2} g + 2\nu t^{\nu-1} g_1 + t^\nu g_2. \end{aligned} \tag{4}$$

By substituting (3) and (4) into (2), we obtain the equation as

$$\begin{aligned} &t g_2 + (\alpha t + 2\nu + 4) g_1 \\ &\quad + [\beta t + (\nu+2)\alpha + (\nu^2 + 3\nu + 2)t^{-1}] g = 0, \end{aligned} \tag{5}$$

If we choose that $\nu^2 + 3\nu + 2 = 0$ in Eq. (5), then, we have $\nu = -2$ and $\nu = -1$. If we write the value of -2, we obtain Eq. (1).

Let $\nu = -1$. So,

$$f = t^{-1} g,$$

and,

$$t g_2 + (\alpha t + 2) g_1 + (\beta t + \alpha) g = 0.$$

After, we obtain

$$\begin{aligned} &t h_2 + [(\alpha + 2\lambda)t + 2] h_1 \\ &\quad + [(\lambda^2 + \alpha\lambda + \beta)t + \alpha + 2\lambda] h = 0, \end{aligned} \tag{6}$$

by using the another transformation as

$$g = e^{\lambda t} h \quad (t \neq 0, h = h(t)).$$

If we choose that $\lambda^2 + \alpha\lambda + \beta = 0$ in Eq. (6), then, we have $\lambda = (-\alpha \pm \delta)/2$, where $\delta = \sqrt{\alpha^2 - 4\beta}$.

Let $\lambda = (\delta - \alpha)/2$. Therefore,

$$g = \mathrm{e}^{\left(\frac{\delta-\alpha}{2}\right)t}h,$$

and,

$$th_2 + (\delta t + 2)h_1 + \delta h = 0. \tag{7}$$

By applying N-fractional calculus method to the both sides of (7), we have

$$(th_2)_\mu + [(\delta t + 2)h_1]_\mu + (\delta h)_\mu = 0.$$

Moreover, we have

$$(th_2)_\mu = th_{2+\mu} + \mu h_{1+\mu}, \tag{8}$$

and,

$$(th_1)_\mu = th_{1+\mu} + \mu h_\mu. \tag{9}$$

By using (8) and (9), we obtain the equation as

$$th_{2+\mu} + (\mu + \delta t + 2)h_{1+\mu} + \delta(\mu + 1)h_\mu = 0. \tag{10}$$

If we suppose that $\mu + 1 = 0$ in Eq. (10), then, we have $\mu = -1$. Therefore, we have a first-order homogeneous linear ordinary differential equation as follows

$$h_1 + (\delta + t^{-1})h = 0. \tag{11}$$

Solution of the Eq. (11) is

$$h = A\mathrm{e}^{-\delta t}t^{-1},$$

where A is an arbitrary constant. So, we have a solution of Eq. (1) as

$$x^{\mathrm{I}}(t) = A\mathrm{e}^{-\left(\frac{\delta+\alpha}{2}\right)t} = A\mathrm{e}^{-\left(\frac{\sqrt{\alpha^2-4\beta}+\alpha}{2}\right)t}.$$

By applying the similar steps, we have another solution of Eq. (1) as

$$x^{\mathrm{II}}(t) = B\mathrm{e}^{\left(\frac{\delta-\alpha}{2}\right)t} = B\mathrm{e}^{\left(\frac{\sqrt{\alpha^2-4\beta}-\alpha}{2}\right)t},$$

where B is an arbitrary constant.

Finally, we obtain the general solution of Eq. (1) as follows

$$x(t) = A\mathrm{e}^{-\left(\frac{\sqrt{\alpha^2-4\beta}+\alpha}{2}\right)t} + B\mathrm{e}^{\left(\frac{\sqrt{\alpha^2-4\beta}-\alpha}{2}\right)t}. \tag{12}$$

The solution given by Eq. (12) is provided for the overdamped system ($\alpha^2 > 4\beta$). In the critically damped system, the solution is defined by

$$x(t) = \mathrm{e}^{-\frac{\alpha}{2}t}(A + Bt),$$

where $\alpha^2 = 4\beta$. If $\alpha^2 < 4\beta$, the system is called underdamped system. So,

$$x(t) = \mathrm{e}^{-\frac{\alpha}{2}t}[A\cos(\omega_d t) + B\sin(\omega_d t)],$$

$$\left(\omega_d = \frac{\sqrt{4\beta - \alpha^2}}{2}\right),$$

where ω_d is natural frequency of the system.

Example 3.1. Let $\boldsymbol{m = 1}$ **kg**, $\boldsymbol{c = 6}$ **kg/s**, $\boldsymbol{k = 25}$ **N/m** in a mass-spring-damper system. Therefore, we have equation as

$$x_2 + 6x_1 + 25x = 0. \tag{13}$$

For Eq. (13), we write

$$\alpha^2 = 36\ \mathrm{s}^{-2} < 4\beta = 100\ \mathrm{s}^{-2},$$

and,

$$\omega_d = \frac{\sqrt{100 - 36}}{2} = 4.$$

So, we obtain the solution of Eq. (13) as follows

$$x(t) = \mathrm{e}^{-3t}(A\cos 4t + B\sin 4t).$$

If $x(0) = 0$, $x_1(0) = 2$ m/s, we have a particular solution of Eq. (13) as

$$x(t) = \frac{1}{2}\mathrm{e}^{-3t}\sin 4t,$$

where $x(t)$ is displacement of the mass time-dependent.

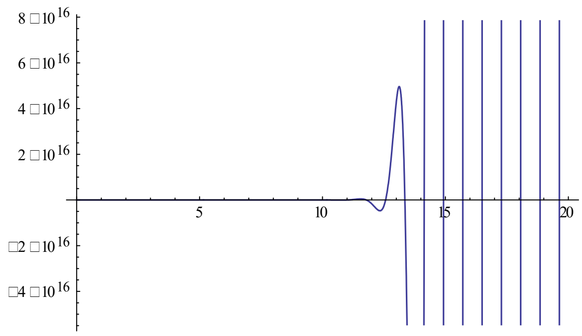

Figure 2. Solution of the system for $t \in [0,20]$.

4. Conclusion

The damped free vibration equation can be solved by means of known methods. In this paper, we studied to find the solutions with a different perspective. We transformed the damped free vibration equation to a singular equation. And, singular differential equations can be solved by means of N-fractional calculus method. So, we used the N-fractional calculus method for this equation. In other words, this method can be applied to linear homogeneous differential equations with constant coefficients.

5. Acknowledgment

The author would like to thank the referees for useful and improving comments.

References

[1] Podlubny I, ‘‘Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, Methods of Their Solution and Some of Their Applications’’, Mathematics in Science and Enginering, USA: Academic Press, 1999.

[2] Yilmazer R, Ozturk O, ‘‘Explicit Solutions of Singular Differential Equation by means of Fractional Calculus Operators’’, Abstract and Applied Analysis, vol. 2013, 6 pages, 2013.

[3] Tu S.T, Chyan D.K, Srivastava H.M, ‘‘Some Families of Ordinary and Partial Fractional Differintegral Equations’’, Integral Transform. Spec. Funct. vol. 11, p.291-302, 2001.

[4] Nishimoto K, ''Kummer's Twenty-Four Functions and N-Fractional Calculus'', NonlinearAnalysis, Theory, Methods & Applications, vol. 30, p.1271-1282, 1997.

[5] Yilmazer R, ''N-Fractional Calculus Operator N^{μ} method to a Modified Hydrogen Atom Equation'', Math. Commun., vol. 15, p.489-501, 2010.

[6] Whittaker E.T, Watson G.N, ''A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions (Fourth Edition)'', Cambridge: Cambridge University Press, 1927.

[7] Fukuhara M, ''Ordinary Differential Equations'', Tokyo: Iwanami Shoten, 1941.

[8] Tricomi F.G, ''Funzioni Ipergeometriche Confluenti'', Roma: Edizioni Cremonese, 1954.

[9] Watson G.N, ''A Treatise on the Theory of Bessel Functions (Second Edition)'', Cambridge: Cambridge University Press, 1944.

[10] Feldman M, ''Non-linear System Vibration Analysis Using Hilbert Transform'', Mechanical Systems and Signal Processing. vol 8 (2), p.119-127, 1994.

[11] Bagley R.L, Torvik J, ''Fractional Calculus - A Different Approach to the Analysis of Viscoelastically Damped Structure'', AIAA Journal. vol. 21 (5), p.741-748, 1983.

[12] Gough C, ''The nonlinear free vibration of a damped elastic string'', J. Acoust. Soc. Am. 75, 1984.

[13] Rossikhin Y, Shitikova M, ''Application of Fractional Derivatives to the Analysis of Damped Vibrations of Viscoelastic Single Mass Systems'', Acta Mechanica vol. 120 (1), p.109-125, 1997.

[14] Rossikhin Y, Shitikova M, ''Application of Fractional Calculus for Analysis of Nonlinear Damped Vibrations of Suspension Bridges'', J. Eng. Mech. vol. 124 (9), p.1029-1036, 1998.